

Diffusion Weighted Imaging as qt-Space Tomography

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Introduction:

Statistical imaging of microscopic motions using MRI has followed two analytical paradigms: diffusion-weighting and q-space analyses. In diffusion-weighted analyses, the ensemble of molecules is assumed to evolve with a Gaussian probability density/point spread function (PSF)—or a mixture of such densities—and the data analysis is oriented towards determining the parameters of the PSF. In q-space analyses, the ensemble of molecules is allowed to have a quite general PSF; as a result, much more data is required to reconstruct the PSF. The two analytical paradigms come from two different experimental communities with little overlap, since diffusion-weighted imaging is practicable in humans, whereas accurate q-space imaging is not. Our goal is to generalize [1], which relates q-space and diffusion weighting.

Analysis:

We denote by $M(\mathbf{x}, t)$ the transverse magnetization and by $\hat{M}(\mathbf{q}, t)$ its spatial Fourier transform. In the absence of magnetic field gradients, we model the transport of magnetization from its initial state $M(\mathbf{x}, 0)$ to time t by convolution with the unknown time-dependent PSF $P(\mathbf{x}, t)$:

$$M(\mathbf{x}, 0) \xrightarrow{t} M(\mathbf{x}, 0) * P(\mathbf{x}, t) \Rightarrow \hat{M}(\mathbf{q}, 0) \xrightarrow{t} \hat{M}(\mathbf{q}, 0) \cdot \hat{P}(\mathbf{q}, t) \Rightarrow \frac{\partial \hat{M}(\mathbf{q}, t)}{\partial t} = \frac{\partial \hat{P}(\mathbf{q}, t)}{\partial t} \cdot \hat{M}(\mathbf{q}, t)$$

Define $\hat{P}(\mathbf{q}, t) = e^{-u(\mathbf{q}, t)}$ (for diffusion, $u(\mathbf{q}, t) = \mathbf{q} \cdot \mathbf{D} \cdot \mathbf{q} t$, where \mathbf{D} is the diffusion tensor to be estimated), so that $\hat{P}(\mathbf{q}, t)^{-1} \cdot \partial \hat{P}(\mathbf{q}, t) / \partial t = -\partial u(\mathbf{q}, t) / \partial t \equiv -u_t(\mathbf{q}, t)$. With gradients, $d\mathbf{q} / dt = \gamma \mathbf{G}(t)$, and the magnetization transport model is:

$$\frac{\partial M(\mathbf{x}, t)}{\partial t} = -i \frac{d\mathbf{q}}{dt} \cdot \mathbf{x} M(\mathbf{x}, t) + \mathfrak{S}_{\mathbf{q} \leftrightarrow \mathbf{x}}^{-1} [-u_t(\mathbf{q}, t) \hat{M}(\mathbf{q}, t)] \Rightarrow \frac{\partial \hat{M}(\mathbf{q}, t)}{\partial t} - \frac{d\mathbf{q}}{dt} \cdot \nabla_{\mathbf{q}} \hat{M}(\mathbf{q}, t) = -u_t(\mathbf{q}, t) \hat{M}(\mathbf{q}, t)$$

The last equation is a first order PDE in (\mathbf{q}, t) space, which can be solved using the method of characteristics:

$\hat{M}(\mathbf{q}_0 - \mathbf{q}(t), t) = e^{-\int_0^t u_t(\mathbf{q}(\tau), \tau) d\tau} \hat{M}(\mathbf{q}_0, 0)$, where \mathbf{q}_0 is an arbitrary vector in q-space. For imaging purposes, the trajectory $\mathbf{q}(t)$ is rewound to $\mathbf{q} = \mathbf{0}$ at some time T before the k-space readout begins. Assuming that the k-space region covered is small enough not to induce significant diffusive effects itself, then we find that the attenuation of the image is given by

$E = e^{-\int_0^T u_t(\mathbf{q}(t), t) dt}$ or $-\ln(E) = \int_0^T u_t(\mathbf{q}(t), t) dt = -\int_0^T [\hat{P}(\mathbf{q}(t), t)^{-1} \cdot \partial \hat{P}(\mathbf{q}(t), t) / \partial t] dt$; that is, a general trajectory through qt-space gives a tomographic result about the time evolution of the Fourier transform of the PSF for water transport.

Discussion: qt-Space Tomography:

In the diffusion limit, $\ln(E) = -\int_0^T \mathbf{q}(t) \cdot \mathbf{D} \cdot \mathbf{q}(t) dt$, the basis for estimating the diffusion tensor \mathbf{D} by using different paths through qt-space. In the PGSE experiment, $T = \Delta$ and $\mathbf{q}(t) = \text{const}$ (since the pulsed gradient duration δ is assumed small), yielding $E(\mathbf{q}) = e^{-\int_0^\Delta u_t(\mathbf{q}, t) dt} = e^{-u(\mathbf{q}, \Delta)} = \hat{P}(\mathbf{q}, \Delta)$, the usual q-space imaging attenuation [1]. If δ is *not* small, then $\mathbf{q}(t) \neq \text{const}$ and the relationship between E and $\hat{P}(\mathbf{q}, t)$ is more complicated and involves all intermediate times $0 \leq t \leq \Delta$. If enough different trajectories through qt-space were traversed, and a parameterized mathematical model for $\hat{P}(\mathbf{q}, t)$ adopted (cf. [2,3]), then the parameters of $\hat{P}(\mathbf{q}, t)$ could be estimated from the $-\ln(E)$ measurements. Such a technique may allow a systematic (if approximate) extension of q-space imaging to humans, where δ cannot be small. Various extensions of this theory are possible, such as allowing for transport effects during excitation and for multiple coherent pathways through qt-space created by multiple RF pulses [4].

References:

- [1] Bassar PJ, *MRM* **47**: 392-397, 2002.
- [2] Özarslan E and Mareci TH, *MRM* **50**: 955-965, 2003.
- [3] Sato K-I, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge, 1999.
- [4] Frank LR, Wong EC, Liu TT, and Buxton RB, *MRM* **49**: 1098-1105, 2003.

Possible trajectories through qt-space, limited by maximum gradient strength

