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Rotation of NMR Images Using the 2D Chirp-*z* Transform

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A quick and accurate way to rotate and shift nuclear magnetic resonance (NMR) images using the two-dimensional chirp-*z* transform is presented. When the desired image grid is rotated and shifted from the original grid due to patient motion, the chirp-*z* transform can reconstruct NMR images directly onto the ultimate grid instead of reconstructing onto the original grid and then applying interpolation to get the final real-space image in the conventional way. The rotation angle and shift distances are embedded in the parameters of the chirp-*z* transform. The chirp-*z* transform implements discrete sinc interpolation to get values at grid points that are not exactly on the original grid when applying the inverse Fourier transform. Therefore, the chirp-*z* transform is more accurate than methods such as linear or bicubic interpolation and is more efficient than direct implementation of sinc interpolation because the sinc interpolation is implemented at the same time as reconstruction from *k*-space. Magn Reson Med 41:253–256, 1999. © 1999 Wiley-Liss, Inc.

Key words: chirp-*z* transform; nuclear magnetic resonance; linear interpolation; bicubic interpolation; sinc interpolation

It is often desirable to rotate and shift nuclear magnetic resonance (NMR) images from the original grid due to patient motion. The interpolation necessary to carry out the rotation and shift can cause the images to be smeared. It is necessary to find a way to reconstruct the image onto the desired grid quickly and with minimal loss of image quality. In the usual method, the images are first reconstructed from *k*-space onto the original grid using the inverse discrete Fourier transform, and then the reconstructed images are interpolated onto the rotated and shifted grid to get the final image. Among all the interpolation methods, since interpolation gives the best result for band-limited data (1). However, sinc interpolation in image-space is difficult to implement efficiently, which restricts its use in MRI. By reconstructing and doing since interpolation together, the two-dimensional (2D) chirp-*z* transform presented here can rotate images more efficiently and accurately than the usual interpolation methods. In a sense, the method presented here is a generalization of the well-known Fourier shift theorem, which allows the computation of shifted images by modifying the phase of the data (2).

THEORY

The 1D chirp-*z* transform is defined on the N-long complex sequence $\{f_j, j = 0, 1, \ldots, N-1\}$ as

$$\tilde{f}_k(\alpha) = \sum_{j=0}^{N-1} f_j\, e^{-2\pi i \cdot jk\alpha}, \qquad [1]$$

where α can be any complex number (3,4). The chirp-*z* transform is a generalization of the discrete Fourier transform (DFT), which is defined as

$$\hat{f}_k = \sum_{j=0}^{N-1} f_j\, e^{-2\pi i \cdot jk/N}.$$

The DFT is a special case of the chirp-*z* transform with the parameter α fixed at $1/N$.

The difference between the DFT and the chirp-*z* transform will be clear if their image-space grids are shown (Fig. 1). The chirp-*z* transform has arbitrary spacing in its image-space grid, given as α, while the spacing in the DFT is fixed as $1/N$. The DFT cannot adjust its grid spacing to fit a different output spacing while the chirp-*z* transform can stretch or shrink the output grid spacing.

This advantage of the chirp-*z* transform over the DFT can be extended to 2D transforms, which means that by using the 2D chirp-*z* transform the relationship between the *k*-space MRI data grid and the reconstructed image-space grid can be arbitrary. Not only can the grid spacing be stretched or shrunk, but also the orientation of the whole grid can be changed. Define the 2D chirp-*z* transform on the 2D complex sequence $\{f_{lm}: l, m = 0, 1, \ldots, N-1\}$:

$$\tilde{f}_{pq}(\alpha, \beta) = \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} f_{lm} e^{-2\pi i\alpha\cdot(lp+mq)} e^{-2\pi i\beta\cdot(mp-lq)}, \qquad [2]$$

where α and β are arbitrary complex numbers. The 2D DFT is defined on the same 2D complex sequence as

$$\hat{f}_{pq} = \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} f_{lm} e^{-2\pi i(lp+mq)/N}.$$

Compared with the 2D chirp-*z* transform, the 2D DFT is the special case of the 2D chirp-*z* transform with $\alpha = 1/N$, $\beta = 0$. Note the third factor of the summation element in Eq. [2]. The grid indices of image-space (p, q) and *k*-space (l, m) are cross multiplied. In a later section we will show how α and β are related to the rotation angle and grid spacing.

ALGORITHM

If the 2D chirp-*z* transform is implemented strictly according to Eq. [2], it will be much slower than the fast Fourier transform (FFT) implementation of the DFT (5). Rabiner et al introduced a 1D fast chirp-*z* transform algorithm (3). This algorithm can be derived by rewriting $2jk = j^2 + k^2 - (k-j)^2$ in Eq. [1] (4). The expression for the 1D chirp-*z*

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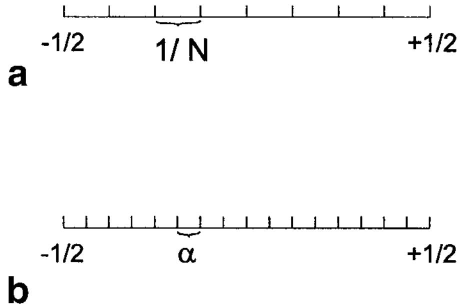

FIG. 1. **a:** The image-space grid for the DFT. **b:** The image-space grid for the chirp-*z* transform.

transform becomes

$$\tilde{f}_k(\alpha) = \sum_{j=0}^{N-1} f_j\, e^{-\pi i[j^2+k^2-(k-j)^2]\alpha}$$

$$= e^{-\pi i k^2\alpha} \sum_{j=0}^{N-1} f_j \cdot e^{-\pi i j^2\alpha} \cdot e^{\pi i(k-j)^2\alpha}. \qquad [3]$$

The summation in Eq. [3] is the convolution of the sequence $\{f_j e^{-\pi i j^{2\alpha}}\}$ with $\{e^{\pi i j^{2\alpha}}\}$. Since the Fourier transform of the convolution of two sequences is equal to the product of the Fourier transform of the sequences, the 1D chirp-z transform can be implemented by the use of the 1D FFT algorithm. We first FFT the sequences $\{f_j e^{-\pi i j^{2\alpha}}\}$ and $\{e^{\pi i j^{2\alpha}}\}$, multiply the results, and apply the inverse FFT to this product. The final step is to multiply this inverse FFT result by the factor $e^{-\pi i k^2\alpha}$.

In the 2D case, which is done here for the first time to the best of our knowledge, factor the exponents in Eq. [2] like this:

$$2(lp + mq) = (l + p)^2 - (m - q)^2 + (m^2 - l^2) + (q^2 - p^2)$$

and

$$(mp - lq) = (l + p)(m - q) - (lm) + (pq),$$

then

$$\tilde{f}_{pq}(\alpha,\beta) = \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} f_{lm} \cdot e^{-\pi i\alpha(m^2-l^2)} \cdot e^{-\pi i\alpha(q^2-p^2)}$$

$$\cdot\, e^{-\pi i\alpha[(l+p)^2-(m-q)^2]} \cdot e^{2\pi i\beta(lm)} \cdot e^{-2\pi i\beta(pq)} \cdot e^{-2\pi i\beta(l+p)(m-q)}.$$

Define

$$Y_{lm} = f_{lm} \cdot e^{-\pi i\alpha\cdot(m^2-l^2)+2\pi i\beta\cdot(lm)}$$

$$Z_{lm} = e^{-\pi i\alpha\cdot(l^2-m^2)-2\pi i\beta\cdot(lm)},$$

then

$$\tilde{f}_{pq}(\alpha, \beta) = (Z_{qp}) \cdot \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} Y_{lm} \cdot Z_{l+p,m-q}. \qquad [4]$$

The summation in Eq. [4] is a 2D discrete convolution that can be evaluated with 2D FFTs. This evaluation shows that the 2D chirp-z transform can be implemented relatively efficiently. We note that the coefficients Z_{lm} can be generated recursively using the identities

$$Z_{l+1,m} = Z^*_{l-1,m} \cdot Z^2_{lm} \cdot e^{-2\pi i\alpha},$$

$$Z_{l,m+1} = Z^*_{l,m-1} \cdot Z^2_{lm} \cdot e^{2\pi i\alpha}.$$

This technique avoids computation of many trigonometric functions. In addition, if only a magnitude image is required, the final multiplication by Z_{qp} may be omitted.

RECONSTRUCTION

The NMR signal $S(k_x, k_y)$ is gathered in k-space (6), which is

$$S(k_x, k_y) \propto \int\!\!\int M_\perp(x, y) e^{-2\pi i(k_x x + k_y y)} dx\, dy \; (+ \text{ noise}).$$

Normal reconstruction applies the inverse Fourier transform to give the image as

$$I(x, y) = \int\!\!\int S(k_x, k_y) e^{2\pi i(k_x x + k_y y)} dk_x dk_y\,. \qquad [5]$$

Since the data are discrete, we actually have

$$S_{lm} = S(l\Delta k, m\Delta k) \quad |l|, |m| \le \frac{1}{2} N.$$

The image reconstruction of Eq. [5] is approximated by a sum

$$I(p\Delta x, q\Delta x) \approx I_{pq} = \sum_{l,m} S_{lm} e^{2\pi i(l\Delta k p\Delta x + m\Delta k q\Delta x)}$$

$$= \sum_{l,m} S_{lm} e^{2\pi i(lp+mq)\Delta k\Delta x}.$$

If $\Delta k \Delta x = 2^{-M}$, a power-of-2 FFT can be used for fast reconstruction. As shown in Fig. 1, once the k-space grid is chosen (by programming the image acquisition parameters), the image-space grid is chosen too. However, we would like to break this yoke so that the data can be reconstructed (semi-efficiently) onto an arbitrary rotated grid. We approximate the integral in Eq. [5] by a sum, but evaluated on a rotated grid.

$$I_{pq} = I(p \cos\theta\, \Delta x - q \sin\theta\, \Delta x,\, p \sin\theta\, \Delta x + q \cos\theta\, \Delta x)$$

$$= \sum_{lm} S_{lm} e^{2\pi i l\Delta k\cdot(p\cos\theta\Delta x - q\sin\theta\Delta x)} e^{2\pi i m\Delta k\cdot(p\sin\theta\Delta z + q\cos\theta\Delta x)}$$

$$= \sum_{lm} S_{lm}\, e^{2\pi i(lp+mq)\cdot\cos\theta\Delta k\Delta x} e^{2\pi i(-lq+mp)\cdot\sin\theta\Delta k\Delta x}. \qquad [6]$$

Eq. [6] requires a way to compute $\Sigma_{lm} S_{lm} e^{2\pi i\alpha\cdot(lp+mq)} e^{2\pi i\beta\cdot(-lq+mp)}$ efficiently for arbitrary α and β, not just $\alpha = 1/N$ and $\beta = 0$. Eq. [2] shows that the chirp-z transform is the ideal tool for this task.

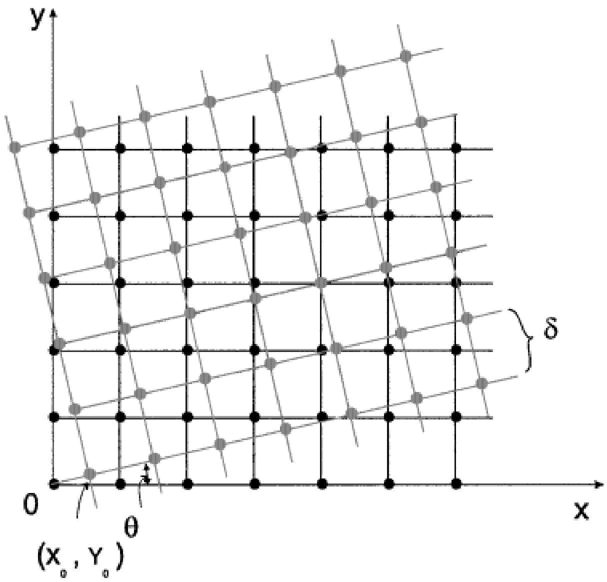

FIG. 2. The image-space grid is rotated with angle θ and shifted with the original point shifted from (0, 0) to (x_0, y_0).

In the rest of this paper the data in k-space are represented by f_{lm}, while $\tilde{f}_{lm}$ stands for the reconstruction on the image-space grid; (l, m) is the input index in k-space with the coordinates of the corresponding point given as $(l\Delta k_x, m\Delta k_y)$; (p, q) is the output index in the image-space with the coordinates of the corresponding point given as (x_{pq}, y_{pq}). If the image-space grid is rotated with angle θ and shifted with the origin point being moved from (0, 0) to (x_0, y_0) (shown in Fig. 2), the value of x_{pq} and y_{pq} is

$$x_{pq} = x_0 + (\Delta x \cos \theta)p - (\Delta x \sin \theta)q, \quad p = 0, 1, \ldots, N - 1.$$

$$y_{pq} = y_0 + (\Delta x \sin \theta)p + (\Delta x \cos \theta)q, \quad q = 0, 1, \ldots, N - 1.$$

where Δx is the spacing in the image-space grid. The image-space reconstruction $\tilde{f}(x, y)$ at position (x, y) is given by

$$\tilde{f}(x, y) = \sum_{l=-(N/2)}^{(N/2)-1} \sum_{m=-(N/2)}^{(N/2)-1} f_{lm} e^{2\pi i l x/N + 2\pi i m y/N}.$$

The inverse DFT cannot be applied directly to get $\tilde{f}(x, y)$. Instead, the value at every grid point (x_{pq}, y_{pq}) can be calculated as:

$$\begin{aligned}
\tilde{f}_{pq} &= \tilde{f}(x_{pq}, y_{pq}) \\
&= \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} f_{l-(N/2),m-(N/2)} \cdot e^{2\pi i \cdot [l-(N/2)](x_0+\Delta x \cos\theta \cdot p - \Delta x \sin\theta \cdot q)/N} \\
&\quad \cdot e^{2\pi i \cdot [m-(N/2)](y_0+\Delta x \sin\theta \cdot p + \Delta x \cos\theta \cdot q)/N} \\
&= \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} f_{l-(N/2),m-(N/2)} \cdot e^{2\pi i [l-(N/2)] x_0/N + 2\pi i [m-(N/2)] y_0/N} \\
&\quad \cdot e^{-\pi i \Delta x [(\cos\theta + \sin\theta) p + (\cos\theta - \sin\theta) q]} \\
&\quad \cdot e^{2\pi i (lp+mq)\cdot \Delta x \cos\theta/N} \cdot e^{2\pi i (mp-lq)\cdot \Delta x \sin\theta/N} \\
&= e^{-\pi i \Delta x [(\cos\theta+\sin\theta) p + (\cos\theta - \sin\theta) q)} \cdot \tilde{H}_{pq}\left(-\frac{\Delta x \cos\theta}{N}, -\frac{\Delta x \sin\theta}{N}\right),
\end{aligned}$$

where $\tilde{H}_{pq}$ is the chirp-z transform of the sequence H_{lm} at indices p, q and

$$H_{lm} = f_{l-(N/2),m-(N/2)}\, e^{2\pi i [(l-(N/2)) x_0 + (m-(N/2)) y_0]/N}.$$

With the chirp-z transform, the image can be efficiently reconstructed directly onto the rotated and shifted grid. From Eq. [4], $\tilde{H}_{pq}$ can be computed with two 2D FFTs of size $2L$ $(L \geq N)$ with the initialization of Z_{lm}. The arithmetic complexity of the chirp-z transform is $\mathcal{O}(80N^2 \log_2 N)$ (usually $L = N$). A direct image-space image-space implementa-

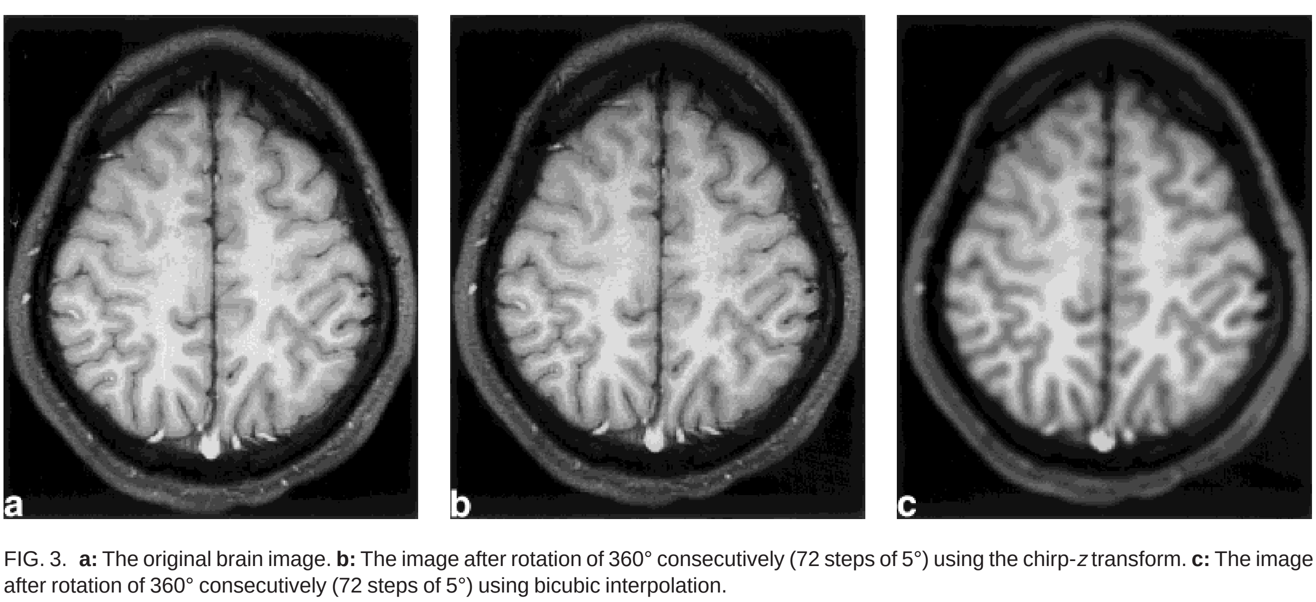

FIG. 3. **a:** The original brain image. **b:** The image after rotation of 360° consecutively (72 steps of 5°) using the chirp-*z* transform. **c:** The image after rotation of 360° consecutively (72 steps of 5°) using bicubic interpolation.

tion of bicubic interpolation with four neighboring points in each direction requires approximately $\mathcal{O}(75N^2)$ operations.

RESULTS AND DISCUSSION

We applied the chirp-z transform to rotate complex-valued MR images (matrix 256×256). As an example, the chirp-z transform can rotate images efficiently with little blurring even after rotating 360° consecutively (72 steps of 5°) compared with bicubic interpolation (shown in Fig. 3). The fidelity of the chirp-z transform rotated images is much higher than the image rotated using bicubic interpolation. Interpolation in k-space can help solve the problem of rotating MR images, but these methods have limitations (7–9). Unser et al and Eddy et al described improved methods to rotate images by using a shearing transformation and Fourier interpolation (10,11). The method presented here is a direct way to rotate MR images at the same time as reconstructing the data from k-space, avoiding any explicit interpolation. Instead of two steps, reconstruction followed by interpolation, the chirp-z transform does the Fourier transform and simultaneously implements discrete sinc interpolation to interpolate the values for the points that are not exactly on the original image-space grid. The chirp-z transform is the ideal method to rotate NMR images quickly and efficiently, since it is the computational realization of Eq. [5]. Combined with an algorithm for determining the rotation angle and shift, we expect this to become a very useful and efficient tool in MRI.

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